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WEYL, PROJECTIVE AND CONFORMAL SEMI-SYMMETRIC COMPLEX HYPERSURFACES IN SEMI-KAEHLER SPACE FORMS

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Dedicated to the memory of Witold Roter

Abstract. The purpose of this paper is to introduce the notions of Weyl semi-symmetric, projective semi-symmetric and conformal semi-symmetric curvature tensor defined on semi-Kaehler manifolds. Moreover, by using a new version of E. Cartan's complex exterior derivative method we give a complete classification of complex hypersurfaces M in semi-Kaehler space forms $M^{n+1}_{s+t}(c)$ with Weyl semi-symmetric, projective semi-symmetric or conformal semi-symmetric curvature tensor, respectively.

1. Introduction. There exist well-known curvature-like tensors associated with various geometric structures on manifolds, analogous to the Riemannian curvature tensor R defined on a Riemannian manifold. Examples are provided by the concircular, projective and conformal curvature tensors (see Mantica and Suh [MS1]–[MS4], Roter [Rt1], [Rt2], Yano and Bochner [YB]).

Besse's book [Be] mentions three curvature-like tensors defined on Kaehler manifolds, namely, the Weyl curvature tensor, the projective curvature tensor and the conformal curvature tensor.

Let M be a complex n-dimensional semi-Kaehler manifold of index $2s$, $0 \leq s \leq n$, with semi-Kaehler connection ∇. We denote by TM the tangent bundle of M. Let T^CM be the complexification of TM. Let T be a quadrilinear mapping of $T^CM \times T^CM \times T^CM \times T^CM$ into $\mathbb{C}$ satisfying the curvature-like conditions

(a) $\bar{T}(X,Y,Z,U) = T(\bar{X},\bar{Y},\bar{Z},\bar{U})$,
(b) $T(JX,JY,Z,U) = T(X,Y,JZ,JU) = T(X,Y,Z,U)$.

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Then T is said to be a *curvature-like tensor* on M. We will show that the Weyl curvature tensor, the projective curvature tensor and the conformal curvature tensor are curvature-like tensors on a semi-Kaehler manifold M.

First, as a complex version of the concircular curvature tensor, on a semi-Kaehler manifold M we introduce a curvature-like tensor W, called the *Weyl curvature tensor*, defined by

$$W_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{ik}\delta_{jl})$$

where R and r denote the curvature tensor and the scalar curvature respectively.

Next, as a complex version of the projective curvature tensor, on a semi-Kaehler manifold M we consider another kind of curvature-like tensor G, called the *complex projective curvature tensor*, defined by

$$G_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(\epsilon_j\delta_{ji}S_{k\bar{l}} + \epsilon_k\delta_{ki}S_{j\bar{l}}),$$

where S denotes the Ricci tensor of M.

Finally, let H be the *conformal curvature tensor* with components $H_{\bar{i}jk\bar{l}}$ defined by

$$H_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}(\varepsilon_k\delta_{kl}S_{j\bar{i}} + \varepsilon_j\delta_{jl}S_{k\bar{i}} + \varepsilon_j\delta_{ij}S_{k\bar{l}} + \varepsilon_k\delta_{ik}S_{j\bar{l}}).$$

This was introduced by Bochner [Bo] as a formal analogue to the Weyl conformal curvature tensor on a Riemannian manifold (see also Yano and Bochner [YB]).

When M is Einstein, its Ricci tensor S satisfies the condition

$$S_{i\bar{j}} = \frac{r}{2n}\epsilon_i\delta_{ij}.$$

In that case the complex projective curvature tensor G is equal to the Weyl curvature tensor W. In Section 6 it is shown that G is the same for two complex connections which are projectively related. Moreover, an indefinite Kaehler manifold with vanishing complex projective curvature tensor G is of constant holomorphic sectional curvature if M is Einstein (see Goldberg [G] and Yano and Bochner [YB]).

Semi-symmetry for semi-Kaehler manifolds was introduced by Choi, Kwon and Suh [CKS1]. A semi-Kaehler manifold M is said to be *semi-symmetric* if it satisfies the condition $R(X,Y)R = 0$ for any vector fields X and Y on M. This condition is weaker than the one characterizing locally symmetric spaces, that is, $\nabla R = 0$. Semi-symmetry for Riemannian spaces was introduced by Szabó [Sz].

In [CKS1] semi-symmetric complex hypersurfaces in semi-Kaehler space forms $M^{n+1}_{s+t}(c)$ are classified as follows:

THEOREM A. *Let M be an n-dimensional semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \leq s \leq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with scalar curvature $r = n(n+1)c$, or Einstein with $r = n^2 c$.*

A semi-Kaehler manifold M is said to be *Weyl semi-symmetric* (resp. *projective semi-symmetric*, *conformal semi-symmetric*) if the Weyl curvature tensor W (resp. projective curvature tensor G, conformal curvature tensor H) satisfies the condition $R(X,Y)W = 0$ (resp. $R(X,Y)G = 0$, $R(X,Y)H = 0$) for any vector fields X and Y on M.

Now we introduce the notion of recurrent curvature-like tensors on semi-Kaehler manifolds (see [MS1], [MS2] and [SK]).

The Weyl curvature tensor W (resp. the projective curvature tensor G, the conformal curvature tensor H) is said to be *recurrent* if there exists a 1-form α such that $\nabla W = \alpha \otimes W$ (resp. $\nabla G = \alpha \otimes G$, $\nabla H = \alpha \otimes H$).

In particular, when the 1-form α vanishes identically, the Weyl recurrence condition is reduced to $\nabla W = 0$, that is, M is Weyl symmetric. If M is locally symmetric, that is, $\nabla R = 0$, then its scalar curvature r is constant. So we know that M is then Weyl symmetric, $\nabla W = 0$. Of course, M must then also be Weyl semi-symmetric. For projective recurrence and conformal recurrence, we have the corresponding results.

Moreover, it can be easily seen that a Kaehler manifold M with recurrent Weyl curvature tensor W (resp. projective curvature tensor G, conformal curvature tensor H) satisfies the second Bianchi identity. Then naturally, by using the method given in Sections 5–7 we can see that the recurrent Weyl curvature tensor W (resp. projective curvature tensor G, conformal curvature tensor H) satisfies the Weyl semi-symmetry condition $R(X,Y)W = 0$ (resp. projective semi-symmetry condition $R(X,Y)G = 0$, conformal semi-symmetric condition $R(X,Y)H = 0$) for a Kaehler manifold M.

Thus, the property of Weyl semi-symmetry (resp. projective semi-symmetry, conformal semi-symmetry) is weaker than Weyl recurrence (resp. projective recurrence, conformal recurrence). But it is an open problem whether the properties of semi-symmetry, Weyl semi-symmetry, projective semi-symmetry and conformal semi-symmetry of complex hypersurfaces in semi-Kaehler space forms $M^{n+1}_{s+t}(c)$, $c \neq 0$, are all equivalent or not.

In this connection, by using a new version of E. Cartan's complex exterior derivative method, we give a complete classification of Weyl semi-symmetric complex hypersurfaces of index $2s$ in $M^{n+1}_{s+t}(c)$:

THEOREM 1. *Let M be an n-dimensional Weyl semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \leq s \leq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with scalar curvature $r = n(n+1)c$, or Einstein with $r = n^2 c$.*

From Theorem 1 we see that semi-symmetry is equivalent to Weyl semi-symmetry for a complex hypersurface in a semi-Kaehler space form $M^{n+1}_{s+t}(c)$.

Though the Weyl curvature tensor W, the projective curvature tensor G and the conformal curvature tensor H are mutually different as defined above, surprisingly we will prove analogous results for semi-symmetric Weyl, projective and conformal hypersurfaces in semi-Kaehler space forms $M^{n+1}_{s+t}(c)$, $c \neq 0$. Accordingly, we give a complete classification of projective semi-symmetric and conformal semi-symmetric complex hypersurfaces in a semi-Kaehler space form $M^{n+1}_{s+t}(c)$:

THEOREM 2. *Let M be an n-dimensional projective semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \leq s \leq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with scalar curvature $r = n(n+1)c$, or Einstein with $r = n^2c$.*

THEOREM 3. *Let M be an n-dimensional conformal semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \leq s \leq n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with scalar curvature $r = n(n+1)c$, or Einstein with $r = n^2c$.*

2. Semi-Kaehler manifolds. This section is concerned with local formulas for semi-Kaehler manifolds. Let M be a complex n-dimensional connected indefinite Kaehler manifold of index $2s$ $(n \geq 2,\ 0 \leq s \leq n)$, equipped with a semi-Kaehler metric tensor g. Let $\{U_j\}$ be a local field of unitary frames on an open set in M. Here and below, the small Latin indices $j, k, \ldots$ run from 1 to n. We have $g(U_j, \bar{U}_k) = \epsilon_j \delta_{jk}$, where

$$\epsilon_j = g(U_j, \bar{U}_j) = -1 \text{ or } 1 \quad \text{according to} \quad 1 \leq j \leq s \quad \text{or} \quad s+1 \leq j \leq n.$$

In particular, if g is positive definite, then $g(U_j, \bar{U}_k) = \delta_{jk}$.

Now, let $\{\omega_j\}$ be the dual coframe field to $\{U_j\}$. Then $\{\omega_j\} = \{\omega_1, \ldots, \omega_n\}$ consists of complex-valued 1-forms of type $(1,0)$ on M such that $\omega_j(U_k) = \epsilon_j \delta_{jk}$ and $\{\omega_j, \bar{\omega}_j\} = \{\omega_1, \ldots, \omega_n, \bar{\omega}_1, \ldots, \bar{\omega}_n\}$ are linearly independent. The semi-Kaehler metric g of M can be expressed as $g = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex-valued 1-forms ω_{jk}, which are usually called *complex connection forms* on M, which satisfy the structure equations

$$d\omega_i + \sum_k \epsilon_k \omega_{ik} \wedge \omega_k = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \qquad \Omega_{ij} = \sum_{k,l} \epsilon_{kl} R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.2}$$

where $\epsilon_{k\cdots l} = \epsilon_k \cdots \epsilon_l$ and Ω_{ij} (resp. $R_{\bar{i}jk\bar{l}}$) denote the components of the Riemannian curvature form (resp. the Riemannian curvature tensor R) of M

(see Barros and Romero [BR], Romero and Suh [RS], Kobayashi and Nomizu [KN]).

The second equation of (2.1) accounts for the skew-Hermitian symmetry of Ω_{ij}, which is equivalent to the symmetry conditions

$$R_{\bar{i}jk\bar{l}} = \bar{R}_{\bar{j}il\bar{k}}.$$

Moreover, by the exterior differentiation of the first and the third equations of (2.1), the first Bianchi identity

$$\sum_j \epsilon_j \Omega_{ij} \wedge \omega_j = 0 \tag{2.3}$$

is obtained. It implies the further symmetry relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}jk\bar{i}} = R_{\bar{l}kj\bar{i}}. \tag{2.4}$$

Now, with respect to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$S = \sum_{i,j} \epsilon_{ij}(S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j), \tag{2.5}$$

where $S_{i\bar{j}} = \sum_k \epsilon_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{j\bar{i}}$. The scalar curvature r of M is also given by

$$r = 2\sum_j \epsilon_j S_{j\bar{j}}. \tag{2.6}$$

The semi-Kaehler manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{i\bar{j}} = \frac{r\epsilon_i\delta_{ij}}{2n}, \qquad S = \frac{r}{2n}g. \tag{2.7}$$

The components $R_{\bar{i}jk\bar{l}n}$ and $R_{\bar{i}jk\bar{l}\bar{n}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are defined by

$$\begin{aligned}\sum_n \epsilon_n(R_{\bar{i}jk\bar{l}n}\omega_n + R_{\bar{i}jk\bar{l}\bar{n}}\bar{\omega}_n) &= dR_{\bar{i}jk\bar{l}} \\ &\quad - \sum_n \epsilon_n(R_{\bar{i}jk\bar{l}}\bar{\omega}_{ni} + R_{\bar{i}nk\bar{l}}\omega_{nj} + R_{\bar{i}jn\bar{l}}\omega_{nk} + R_{\bar{i}jk\bar{n}}\bar{\omega}_{nl}),\end{aligned} \tag{2.8}$$

$$\sum_k \epsilon_k(S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k \epsilon_k(S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{2.9}$$

The second Bianchi identity

$$d\Omega_{ij} = \sum_k \epsilon_k(\Omega_{ik} \wedge \omega_{kj} - \omega_{ik} \wedge \Omega_{kj}) \tag{2.10}$$

arises from the exterior differentiation of the first of the structure equations (2.2). In fact,

$$\begin{aligned} d\Omega_{ij} &= \sum_k \epsilon_k d(\omega_{ik}\wedge\omega_{kj}) = \sum_k \epsilon_k (d\omega_{ik}\wedge\omega_{kj} - \omega_{ik}\wedge d\omega_{kj}) \\ &= \sum_j \Big\{ \Big(\Omega_{ik} - \sum_l \epsilon_m \omega_{il}\wedge\omega_{lk}\Big)\wedge\omega_{kj} - \omega_{ik}\wedge\Big(\Omega_{kj} - \sum_l \omega_{kl}\wedge\omega_{lj}\Big)\Big\} \\ &= \sum_k \epsilon_k(\Omega_{ik}\wedge\omega_{kj} - \omega_{ik}\wedge\Omega_{kj}), \end{aligned}$$

where the first equality holds since $d^2 = 0$, the second one follows from the fact that the complex connection form is a 1-form combined with the properties of the exterior derivative, and the third one is derived from the structure equations (2.1).

We can regard $\Omega = (\Omega_{ij})$ and $\omega = (\omega_{ij})$ as complex $n\times n$ matrices. Then (2.10) can be rewritten as

$$d\Omega = \Omega\wedge\omega - \omega\wedge\Omega. \tag{2.11}$$

By straightforward calculation we obtain

$$R_{\bar{i}jk\bar{l}n} = R_{\bar{i}jn\bar{l}k}, \tag{2.12}$$

and hence

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l \epsilon_l R_{\bar{j}ik\bar{l}l}, \quad r_i = 2\sum_k S_{i\bar{k}k}, \tag{2.13}$$

where the exterior differential dr of the scalar curvature r on M is given by

$$dr = \sum_l \epsilon_l (r_l\omega_l + r_{\bar{l}}\bar{\omega}_l). \tag{2.14}$$

Let M be an m-dimensional semi-Kaehler manifold of index $2q$ $(0 \le q \le m)$. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate* provided that $g_x|_P$ is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{X, Y\}$ such that

$$g(X,X)g(Y,Y) - g(X,Y)^2 \ne 0.$$

If the non-degenerate plane P is invariant under the complex structure J, it is said to be *holomorphic*. For the non-degenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The sectional curvature $K(P)$ of the holomorphic plane P is called the *holomorphic sectional curvature*, and denoted by $H(P)$. The semi-Kaehler manifold M is said to be *of constant holomorphic sectional curvature* if $H(P)$ has the same value for all holomorphic planes P at all points of M. Then M is called a *semi-Kaehler space form*, and denoted by $M_q^m(c)$ whenever it is of constant holomorphic sectional curvature c, of complex dimension m and of index $2q$ (≥ 0).

A semi-Kaehler manifold of constant holomorphic sectional curvature is called a *semi-Kaehler space form.* An n-dimensional semi-Kaehler space form of constant holomorphic sectional curvature c and of index $2s$, $0 \leq s \leq n$, is denoted by $M_s^n(c)$. The components $R_{\bar{i}jk\bar{l}}$ of the Riemannian curvature tensor R of $M_s^n(c)$ are given by

$$R_{\bar{i}jk\bar{l}} = c\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})/2. \tag{2.15}$$

3. Semi-Kaehler submanifolds. This section is concerned with semi-Kaehler submanifolds of semi-Kaehler manifolds. First of all, the basic formulas for the theory of semi-Kaehler submanifolds are presented (see Choi, Kwon and Suh [CKS1] and [CKS2], Romero and Suh [RS], Suh and Yang [SY]).

Let M' be an $(n+p)$-dimensional connected semi-Kaehler manifold of index $2(s+t)$ $(0 \leq s \leq n,\ 0 \leq t \leq p)$ with semi-Kaehler structure (g', J'). Let M be an n-dimensional connected semi-Kaehler submanifold of M' and let g be the semi-Kaehler metric tensor of index $2s$ induced on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on an open set in M' in such a way that, restricted to M, $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and below, the following convention on the ranges of indices is used, unless otherwise stated:

$$\begin{aligned} &A, B, C, \ldots = 1, \ldots, n, n+1, \ldots, n+p;\\ &i, j, k, l, \ldots = 1, \ldots, n; \qquad x, y, z, \ldots = n+1, \ldots, n+p. \end{aligned}$$

Let $\{\omega_A\} = \{\omega_j, \omega_y\}$ be the dual frame fields. Then the semi-Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \overline{\omega}_A$, where $\{\varepsilon_A\} = \{\varepsilon_j, \varepsilon_y\}$, $\varepsilon_A = \pm 1$. The connection forms on M' are denoted by $\{\omega_{AB}\}$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\begin{aligned} &d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \overline{\omega}_{BA} = 0,\\ &d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB},\\ &\Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \overline{\omega}_D, \end{aligned} \tag{3.1}$$

where Ω'_{AB} (resp. $R'_{\bar{A}BC\bar{D}}$) denote the components of the curvature form (resp. the Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced semi-Kaehler metric tensor g of index $2s$ on M is given by

$$g = 2\sum_j \varepsilon_j \omega_j \otimes \overline{\omega}_j.$$

Then $\{U_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field to $\{U_j\}$, which consists of complex-valued 1-forms of type $(1,0)$ on M. Moreover $\omega_1,\ldots,\omega_n,\overline{\omega}_1,\ldots,\overline{\omega}_n$ are linearly independent, and $\{\omega_j\}$ are the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivative of (3.2) gives rise to

$$\omega_{xi} = \sum_j \varepsilon_j h_{ij}{}^x \omega_j, \quad h_{ij}{}^x = h_{ji}{}^x. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h_{ij}{}^x \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM of M in M' is called the *second fundamental form* of the submanifold M. The structure equations for M are similarly given by

$$\begin{aligned} & d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \overline{\omega}_{ji} = 0, \\ & d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \qquad \Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \overline{\omega}_l. \end{aligned} \tag{3.4}$$

Moreover,

$$d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \qquad \Omega_{xy} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \overline{\omega}_l, \tag{3.5}$$

where Ω_{xy} is the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M' respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h_{jk}{}^x \overline{h}_{il}{}^x. \tag{3.6}$$

Also, in view of (3.3) and (3.5),

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_r \varepsilon_r h_{kr}{}^x \overline{h}_{rl}{}^y. \tag{3.7}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_k \varepsilon_k R'_{\bar{j}ik\bar{k}} - h_{i\bar{j}}{}^2, \tag{3.8}$$

$$r = 2\Big(\sum_{k,j} \varepsilon_k \varepsilon_j R'_{\bar{k}kj\bar{j}} - h_2\Big), \tag{3.9}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,r} \varepsilon_x \varepsilon_r h_{ir}{}^x \overline{h}_{rj}{}^x$ and $h_2 = \sum_j \varepsilon_j h_{j\bar{j}}{}^2$.

Next, the components ${h_{ijk}}^x$ and ${h_{ij\bar{k}}}^x$ of the covariant derivative of the second fundamental form on M are given by

$$\sum_k \varepsilon_k({h_{ijk}}^x \omega_k + {h_{ij\bar{k}}}^x \overline{\omega}_k) = d{h_{ij}}^x - \sum_k \varepsilon_k({h_{kj}}^x \omega_{ki} + {h_{ik}}^x \omega_{kj}) + \sum_y \varepsilon_y {h_{ij}}^y \omega_{xy}. \tag{3.10}$$

Substituting $d{h_{ij}}^x$ from this definition into the exterior derivative of (3.3) and using (3.1)–(3.4) and (3.8), we have

$${h_{ijk}}^x = {h_{ikj}}^x, \qquad {h_{ij\bar{k}}}^x = -R'_{\bar{x}ij\bar{k}}. \tag{3.11}$$

Similarly the components ${h_{ijkl}}^x$ and ${h_{ijk\bar{l}}}^x$ (resp. ${h_{ij\bar{k}l}}^x$ and ${h_{ij\bar{k}\bar{l}}}^x$) of the covariant derivative of ${h_{ijk}}^x$ (resp. $h_{ij\bar{k}}$) can be expressed as

$$\sum_l \varepsilon_l({h_{ijkl}}^x \omega_l + {h_{ijk\bar{l}}}^x \overline{\omega}_l) = d{h_{ijk}}^x - \sum_l \varepsilon_l({h_{ljk}}^x \omega_{li} + {h_{ilk}}^x \omega_{lj} + {h_{ijl}}^x \omega_{lk}) + \sum_y \varepsilon_y {h_{ijk}}^y \omega_{xy}, \tag{3.12}$$

$$\sum_l \varepsilon_l({h_{ij\bar{k}l}}^x \omega_l + {h_{ij\bar{k}\bar{l}}}^x \overline{\omega}_l) = d{h_{ij\bar{k}}}^x - \sum_l \varepsilon_l({h_{lj\bar{k}}}^x \omega_{li} + {h_{il\bar{k}}}^x \omega_{lj} + {h_{ij\bar{l}}}^x \overline{\omega}_{lk}) + \sum_y \varepsilon_y {h_{ij\bar{k}}}^y \omega_{xy}. \tag{3.13}$$

Taking the exterior derivative of (3.10) and using the properties $d^2 = 0$, (3.4), (3.5), (3.8), (3.10) and (3.11), we have the following Ricci formula for the second fundamental form:

$${h_{ijkl}}^x = {h_{ijlk}}^x, \qquad {h_{ij\bar{k}\bar{l}}}^x = {h_{ij\bar{l}\bar{k}}}^x, \tag{3.14}$$

$${h_{ijk\bar{l}}}^x - {h_{ij\bar{l}k}}^x = \sum_r \varepsilon_r (R_{\bar{l}ki\bar{r}} {h_{rj}}^x + R_{\bar{l}kj\bar{r}} {h_{ir}}^x) - \sum_y \varepsilon_y R_{\bar{x}yk\bar{l}} {h_{ij}}^y. \tag{3.15}$$

In particular, let the ambient space M' be an $(n+p)$-dimensional semi-Kaehler space form $M^{n+p}_{s+t}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$ $(0 \le s \le n,\ 0 \le t \le p)$. Then we get

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}\varepsilon_j \varepsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \varepsilon_x {h_{jk}}^x {\overline{h}_{il}}^x, \tag{3.16}$$

$$S_{i\bar{j}} = \frac{(n+1)c}{2}\varepsilon_i \delta_{ij} - {h_{i\bar{j}}}^2, \tag{3.17}$$

$$r = n(n+1)c - 2h_2, \tag{3.18}$$

$${h_{ij\bar{k}}}^x = 0, \tag{3.19}$$

$$h_{ij k\bar{l}}{}^x = \frac{c}{2}(\varepsilon_k h_{ij}{}^x \delta_{kl} + \varepsilon_i h_{jk}{}^x \delta_{il} + \varepsilon_j h_{ki}{}^x \delta_{jl}) - \sum_{r,y} \varepsilon_r \varepsilon_y (h_{ri}{}^x h_{jk}{}^y + h_{rj}{}^x h_{ki}{}^y + h_{rk}{}^x h_{ij}{}^y)\overline{h}_{rl}{}^y. \tag{3.20}$$

For brevity, a tensor $h_{i\bar{j}}{}^{2m}$ and a function h_{2m} on M for any integer m (≥ 2) are introduced as follows:

$$h_{i\bar{j}}{}^{2m} = \sum_{i_1,\dots,i_{m-1}} \varepsilon_{i_1} \cdots \varepsilon_{i_{m-1}} h_{i\bar{i}_1}{}^2 h_{i_1\bar{i}_2}{}^2 \cdots h_{i_{m-1}\bar{j}}{}^2, \qquad h_{2m} = \sum_i \varepsilon_i h_{i\bar{i}}{}^{2m}.$$

In particular, if M is a hypersurface, then a tensor $h_{ij}{}^{2m+1}$ on M is introduced by

$$h_{ij}{}^{2m+1} = \sum_k \varepsilon_k h_{i\bar{k}}{}^{2m} h_{kj}.$$

4. Examples of indefinite Einstein complex submanifolds. We give some examples of indefinite Einstein submanifolds of an indefinite complex space form:

EXAMPLE 4.1. The indefinite Euclidean space C_s^n of index $2s$ is a totally geodesic complex hypersurface in C_s^{n+1} or C_{s+1}^{n+1} in a natural way.

EXAMPLE 4.2. For an indefinite complex projective space $CP_s^{n+1}(c)$ of index $2s$ and of constant holomorphic sectional curvature c, if $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{n+2}\}$ is the usual homogeneous coordinate system of $CP_s^{n+1}(c)$, then for each fixed j, the equation $z_j = 0$ defines a totally geodesic complex hypersurface identifiable with $CP_s^n(c)$ or $CP_{s-1}^n(c)$ according to whether $s+1 \leq j \leq n+2$ or $1 \leq j \leq s$. By taking into account that $CH_s^n(-c)$ is obtained from $CP_{n-s}^n(c)$ by reversing the sign of its indefinite Kaehler metric, the previous discussion shows that $CH_s^n(-c)$ is a totally geodesic complex hypersurface in both $CH_s^{n+1}(-c)$ and $CH_{s+1}^{n+1}(-c)$ (see Montiel and Romero [MR]).

EXAMPLE 4.3. Let Q_s^n be the indefinite complex hypersurface in $CP_s^{n+1}(c)$ defined by the equation

$$-\sum_{j=1}^{s} z_j{}^2 + \sum_{k=s+1}^{n+2} z_k{}^2 = 0$$

in the homogeneous coordinate system of $CP_s^{n+1}(c)$. Then Q_s^n is a complete indefinite complex hypersurface of index $2s$, and moreover, for reasons similar to those in Kobayashi and Nomizu [KN, Chapter 11, Example 10.6], it is Einstein and its Ricci tensor S satisfies $S = ncg/2$ (see also Romero [R1] and [R3]). This is called an *indefinite complex quadric.*

Note that Q_s^n can also be constructed as an indefinite Einstein complex hypersurface in $CH_{s+1}^{n+1}(-c)$.

EXAMPLE 4.4. Szabó [Sz] showed that a complete Einstein complex hypersurface in a complex space form $M^{n+1}(c)$ is totally geodesic or $c > 0$. In the latter case M is locally congruent to the complex quadric Q^n. In Example 4.3 we can see that the situation of Q^n_s is completely different from those of the definite cases.

REMARK 4.5. Indefinite Einstein complex hypersurfaces in an indefinite complex space form have been investigated in detail by Montiel and Romero [MR] and in Romero's surveys [R2] and [R5].

EXAMPLE 4.6. We consider an indefinite complex hypersurface in $CP^{2n+1}_{n+1}(c)$ defined by the equation

$$\sum_{j=1}^{n+1} z_j z_{n+1+j} = 0$$

in the homogeneous coordinate system of $CP^{2n+1}_{n+1}(c)$. It is a complete complex hypersurface of index $2n$, denoted by $Q^{2n^*}_n$. It is easily seen that its Ricci tensor S satisfies $S = (n+1)cg$, and hence it is Einstein (see [KN]).

5. Weyl semi-symmetric complex hypersurfaces. This section is concerned with Weyl semi-symmetric complex hypersurfaces in a semi-Kaehler space form. Let M be an n-dimensional semi-Kaehler hypersurface of index $2s$ in an $(n+1)$-dimensional semi-Kaehler space form $M' = M^{n+1}_{s+t}(c)$, $0 \le s \le n$, $t = 0$ or 1, of index $2(s+t)$ and of constant holomorphic sectional curvature c. We denote by R the Riemannian curvature tensor on M.

Let W be the Weyl curvature tensor with components $W_{\bar{i}jk\bar{l}}$ defined by

$$W_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - r\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{ki}\delta_{jl})/(2n(n+1)). \tag{5.1}$$

The hypersurface M is said to be *Weyl semi-symmetric* if the Weyl curvature tensor W satisfies

$$R(X,Y)W = 0, \qquad X, Y \in TM. \tag{5.2}$$

It can be easily verified that (5.2) is equivalent to

$$W_{\bar{i}jk\bar{l}m\bar{p}} - W_{\bar{i}jk\bar{l}\bar{p}m} = 0. \tag{5.3}$$

In fact, by applying the Ricci identity to W, we have

$$\begin{aligned} &W_{\bar{i}jk\bar{l}m\bar{p}} - W_{\bar{i}jk\bar{p}l\bar{m}} \\ &\quad = -\sum_r \varepsilon_r (R_{\bar{k}rj\bar{i}} W_{\bar{r}lm\bar{p}} - R_{\bar{r}mj\bar{i}} W_{\bar{k}lr\bar{p}} + R_{\bar{p}rj\bar{i}} W_{\bar{k}lm\bar{r}} - R_{\bar{r}lj\bar{i}} W_{\bar{k}rm\bar{p}}). \end{aligned}$$

For a local unitary frame $\{U_j\}$ on M, the components $R_{\bar{h}ij\bar{k}}$ of the Riemannian curvature tensor R and the components $W_{\bar{h}ij\bar{k}}$ of the Weyl curvature tensor W are given by

$$R(U_i,\overline{U}_j)U_k=\sum_r \varepsilon_r R_{\bar{r}ki\bar{j}}U_r, \qquad R(U_i,\overline{U}_j)\overline{U}_k=\sum_r \varepsilon_r R_{r\bar{k}i\bar{j}}\overline{U}_r,$$
$$W(U_i,\overline{U}_j)U_k=\sum_r \varepsilon_r W_{\bar{r}ki\bar{j}}U_r, \quad W(U_i,\overline{U}_j)\overline{U}_k=\sum_r \varepsilon_r W_{r\bar{k}i\bar{j}}\overline{U}_r.$$

Accordingly, we have

$$\begin{aligned}
&(R(U_j,\overline{U}_i)W)(U_m,\overline{U}_p,U_l)\\
&\quad = R(U_j,\overline{U}_i)W(U_m,\overline{U}_p)U_l - W(R(U_j,\overline{U}_i)U_m,\overline{U}_p)U_l\\
&\qquad - W(U_m,R(U_j,\overline{U}_i)\overline{U}_p)U_l - W(U_m,\overline{U}_p)R(U_j,\overline{U}_i)U_l\\
&\quad = \sum_r \varepsilon_r\{W_{\bar{r}lm\bar{p}}R(U_j,\overline{U}_i)U_r - R_{\bar{r}jl\bar{m}}W(U_r,\overline{U}_p)U_l\\
&\qquad + R_{\bar{p}rj\bar{i}}W(U_m,\overline{U}_r)U_l - R_{\bar{r}lj\bar{i}}W(U_m,\overline{U}_p)U_r\}\\
&\quad = \sum_{r,k}\varepsilon_r\varepsilon_k(W_{\bar{r}lm\bar{p}}R_{\bar{k}rj\bar{i}} - R_{\bar{r}mj\bar{i}}W_{\bar{k}lr\bar{p}} + R_{\bar{p}rj\bar{i}}W_{\bar{k}lm\bar{r}} - R_{\bar{r}lj\bar{i}}W_{\bar{k}rm\bar{p}})U_k.
\end{aligned}$$

So (5.2) is equivalent to

$$\sum_r \varepsilon_r(R_{\bar{k}rj\bar{i}}W_{\bar{r}lm\bar{p}} - R_{\bar{r}mj\bar{i}}W_{\bar{k}lr\bar{p}} + R_{\bar{p}rj\bar{i}}W_{\bar{k}lm\bar{r}} - R_{\bar{r}lj\bar{i}}W_{\bar{k}rm\bar{p}})=0. \tag{5.4}$$

On the other hand, by (3.16), (3.18) and (5.1) we have

$$\begin{aligned}
W_{\bar{i}jk\bar{l}} &= \left\{\frac{c}{2}\varepsilon_{jk}(\delta_{ji}\delta_{kl}+\delta_{ik}\delta_{jl}) - h_{jk}\overline{h}_{il}\right\}\\
&\qquad - \{n(n+1)c-2h_2\}\varepsilon_{jk}(\delta_{ji}\delta_{kl}+\delta_{ik}\delta_{jl})/(2n(n+1))\\
&= -h_{jk}\overline{h}_{il}+\frac{h_2}{n(n+1)}\varepsilon_{jk}(\delta_{ji}\delta_{kl}+\delta_{ik}\delta_{jl}).
\end{aligned} \tag{5.5}$$

By substituting (3.16) and (5.5) into (5.4), we obtain

$$\begin{aligned}
&\sum_r \varepsilon_r\Bigg[-\left\{\frac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rk}+\delta_{ir}\delta_{jk}) - h_{jr}\overline{h}_{ik}\right\}\Bigg\{-h_{lm}\overline{h}_{rp}\\
&\qquad\qquad + \frac{h_2}{n(n+1)}\varepsilon_{lm}(\delta_{rl}\delta_{mp}+\delta_{rm}\delta_{lp})\Bigg\}\\
&\quad + \left\{\frac{c}{2}\varepsilon_{jl}(\delta_{ji}\delta_{lr}+\delta_{il}\delta_{jr}) - h_{jl}\overline{h}_{ir}\right\}\left\{-h_{rm}\overline{h}_{kp}+\frac{h_2}{n(n+1)}\varepsilon_{rm}(\delta_{kr}\delta_{mp}+\delta_{km}\delta_{rp})\right\}\\
&\quad + \left\{\frac{c}{2}\varepsilon_{jm}(\delta_{ji}\delta_{mr}+\delta_{im}\delta_{jr}) - h_{jm}\overline{h}_{ir}\right\}\left\{-h_{lr}\overline{h}_{kp}+\frac{h_2}{n(n+1)}\varepsilon_{lr}(\delta_{kl}\delta_{rp}+\delta_{kr}\delta_{lp})\right\}\\
&\quad - \left\{\frac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rp}+\delta_{ir}\delta_{jp}) - h_{jr}\overline{h}_{ip}\right\}\left\{-h_{lm}\overline{h}_{kr}+\frac{h_2}{n(n+1)}\varepsilon_{lm}(\delta_{kl}\delta_{mr}+\delta_{km}\delta_{lr})\right\}\Bigg]\\
&\qquad = 0,
\end{aligned}$$

which implies that

$$(5.6)\quad 2({h_{j\bar p}}^2\overline{h}_{ik} + {h_{j\bar k}}^2\overline{h}_{ip})h_{lm} - 2({h_{m\bar i}}^2 h_{jl} + {h_{l\bar i}}^2 h_{jm})\overline{h}_{kp} - c(\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip} - \varepsilon_l\delta_{li}h_{jm}\overline{h}_{kp} - \varepsilon_m\delta_{mi}h_{lj}\overline{h}_{kp} + \varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ki}) = 0.$$

THEOREM 5.1. *Let M be an n-dimensional Weyl semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \le s \le n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with $r = n(n+1)c$, or Einstein with $r = n^2c$, where r denotes the scalar curvature.*

Proof. Since $RW = 0$, equation (5.6) holds. Setting $j = p$ in (5.6), multiplying the equation by ε_j and summing over j we get

$$(5.7)\quad 2(h_2\overline{h}_{ik} + {\overline{h}_{ik}}^3)h_{lm} - 2({h_{m\bar i}}^2{h_{l\bar k}}^2 + h_{l\bar i}{h_{m\bar k}}^2) - c\{(n+1)h_{lm}\overline{h}_{ik} - \varepsilon_l\delta_{li}{h_{m\bar k}}^2 - \varepsilon_m\delta_{mi}{h_{l\bar k}}^2\} = 0,$$

Furthermore, setting $k = l$ in (5.7), multiplying the equation by ε_k and summing over k, we get

$$(5.8)\quad c(n{h_{m\bar i}}^2 - h_2\varepsilon_m\delta_{mi}) = 0,$$

which implies that M is Einstein, because of the relation $c \neq 0$ and (3.17).

Next, we investigate the scalar curvature r of M. As ${h_{m\bar i}}^2 = (h_2/n)\varepsilon_m\delta_{mi}$, equation (5.7) is reduced to

$$(5.9)\quad (2h_2 - nc)\{n(n+1)h_{ij}\overline{h}_{ik} - h_2\varepsilon_{lm}(\delta_{mi}\delta_{lk} + \delta_{li}\delta_{mk})\} = 0.$$

Since M is Einstein, h_2 is a constant. So, first consider the case where $2h_2 - nc = 0$ on M, so that the squared norm h_2 of the second fundamental form is $(n/2)c$. Then by using (3.17) and (3.18) we find that M is Einstein with constant scalar curvature $r = n^2c$.

Secondly, assume that $2h_2 - nc \neq 0$ on M. Then (5.9) gives

$$(5.10)\quad n(n+1)h_{ij}\overline{h}_{ik} - h_2\varepsilon_{lm}(\delta_{mi}\delta_{lk} + \delta_{li}\delta_{mk}) = 0.$$

Multiplying (5.10) by $\varepsilon_k h_{kt}$ and summing over k, we get

$$n(n+1){h_{t\bar i}}^2 h_{lm} - h_2(\varepsilon_m\delta_{mi}\delta_{lt} + \varepsilon_l\delta_{li}\delta_{mt}) = 0.$$

Using the above equation and the relation ${h_{m\bar i}}^2 = (h_2/n)\varepsilon_m\delta_{mi}$, we obtain

$$h_2\{(n+1)\varepsilon_l\delta_{ti}h_{lm} - (\varepsilon_m\delta_{mi}h_{lt} + \varepsilon_l\delta_{li}h_{mt})\} = 0.$$

Setting $t = i$, multiplying by ε_t and summing over t gives

$$(n+2)(n-1)h_2h_{lm} = 0.$$

Thus we get $h_2 = 0$ on M, from which by (5.10) it follows that $h_{ij} = 0$ on M. Hence M is totally geodesic with $r = n(n+1)c$, where we have used (3.18). ∎

Now let us recall the following result of Nakagawa and Takagi [NT].

Theorem B. *Let M be a complete Kaehler submanifold imbedded into CP^N with parallel second fundamental form. If M is irreducible, then M is congruent to one of the following Kaehler submanifolds imbedded into CP^N $(N = n + p)$ with parallel second fundamental form:*

$$CP^n = SU(n+1)/S(U(n) \times U(1)), \qquad Q^n = SO(n+2)/SO(n) \times SO(2),$$
$$SU(r+2)/S(U(r) \times U(2)) \ (r \geq 3), \qquad SO(10)/U(5), \qquad E^6/Spin(10) \times T,$$

where $U(n)$, $SU(n)$ and $SO(n)$ denote the unitary group, the special unitary group and the special orthogonal group of order n respectively, and E_6, $Spin(10)$ and T denote the exceptional group, the spin group, and the torus group respectively. If M is reducible, then M is congruent to $(CP^{n_1} \times CP^{n_2}, f)$ for some n_1 and n_2 with $\dim M = n_1 + n_2$, where

$$f : CP^{n_1} \times CP^{n_2} \to CP^{n_1+n_2+n_1n_2}$$

is the Kaehler imbedding. The corresponding local version is also true.

Naturally, a Kaehler submanifold with parallel second fundamental form is locally symmetric. So it is semi-symmetric, and hence by (5.1) and (5.4) it is Weyl semi-symmetric. Now, by using Theorem 5.1 and Theorem B we give a complete classification of Weyl semi-symmetric hypersurfaces in complex projective space:

Theorem 5.2. *Let M be an n-dimensional complex hypersurface in CP^{n+1}. If M is Weyl semi-symmetric, then it is locally congruent to a complex quadric Q^n or to CP^n.*

Proof. More generally, let M be an n-dimensional complex hypersurface in an $(n+1)$-dimensional complex space form $M^{n+1}(c)$, $c \neq 0$. Assume that M is Weyl semi-symmetric. Then ${h_{i\bar{j}}}^2 = h_2\delta_{ij}/n$, and so M is Einstein by Theorem 5.1. Accordingly, the scalar curvature r is constant on M. Then by (3.18) we find that h_2 is constant. Since M is hypersurface, we see that ${h_{i\bar{j}}}^2 = \sum_r h_{ir}\overline{h}_{rj}$.

Differentiating this relation covariantly, by (3.11), (3.19) and the fact that h_2 is constant, we obtain $\sum_r h_{ikr}\overline{h}_{rj} = 0$. Since $h_2 = nc/2$ if M is not totally geodesic, we see that $h_{ijk} = 0$, which means that the second fundamental form of M is parallel.

Combining this result with Theorem B and considering the codimension $p = 1$, we complete the proof. ∎

Also, in the proof of Theorem 5.1, if we consider the two cases concerned with the length of the second fundamental form h_2, we can easily verify the following:

COROLLARY 5.3. *Let M be an n-dimensional complex hypersurface in a complex hyperbolic space $H^{n+1}(c)$, $c < 0$. If M is Weyl semi-symmetric, then it is totally geodesic.*

Proof. When M satisfies $2h_2 = nc$ for $c < 0$, the squared norm $h_2 = \sum_{i,j} h_{ij}\bar{h}_{ij}$ of M in $M^{n+1}(c)$, $c < 0$, cannot be positive definite. This gives us a contradiction. ∎

By using the same method as in the proof of Corollary 5.3 we obtain:

COROLLARY 5.4. *Let M be an n-dimensional space-like complex hypersurface in an indefinite complex space form $M_1^{n+1}(c)$, $c > 0$. If M is Weyl semi-symmetric, then it is totally geodesic.*

6. Projective semi-symmetric complex hypersurfaces. Let M be an n-dimensional complex hypersurface of index $2s$ in an $(n+1)$-dimensional semi-Kaehler space form $M' = M_{s+t}^{n+1}(c)$, $0 \le s \le n$, $t = 0$ or 1, of index $2(s+t)$ and of constant holomorphic sectional curvature c. We choose a local field $\{U_j\}$ of unitary frames on M. Let $\{\omega_j\}$ be the dual frame fields. Then the indefinite Kaehler metric tensor g of M is given by

$$g = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j .$$

The connection forms on M are denoted by $\omega = \{\omega_{ij}\}$.

Let M be an indefinite Kaehler manifold with two indefinite Kaehler metrics g and g'. Then the corresponding connection forms ω and ω' are *projectively related* if there exists a 1-form p such that the coefficients of the connection forms ω and ω' satisfy

$$\omega'_{ij}(U_k) = \omega_{ij}(U_k) + p_j \epsilon_k \delta_{ki} + p_k \epsilon_j \delta_{ji}.$$

It can be easily seen that their Riemannian curvature tensors satisfy

$$R'_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} + \epsilon_j \delta_{ji} p_{k\bar{l}} + \epsilon_k \delta_{ki} p_{j\bar{l}}.$$

The corresponding projective curvature tensors G and G' have components defined by

$$G_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(\epsilon_j \delta_{ji} S_{k\bar{l}} + \epsilon_k \delta_{ki} S_{j\bar{l}}), \tag{6.1}$$

$$G'_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(\epsilon_j \delta_{ji} S'_{k\bar{l}} + \epsilon_k \delta_{ki} S'_{j\bar{l}}). \tag{6.2}$$

By (6.1), we have

$$S'_{k\bar{l}} = \sum_r \epsilon_r R'_{\bar{r}rk\bar{l}} = S_{k\bar{l}} + (n+1)p_{k\bar{l}}, \qquad S'_{j\bar{l}} = S_{j\bar{l}} + (n+1)p_{j\bar{l}}.$$

From (6.1), (6.2) and the above equations, we get

$$\begin{aligned}G'_{\bar{i}jk\bar{l}} &= (R_{\bar{i}jk\bar{l}} + \epsilon_j\delta_{ji}p_{k\bar{l}} + \epsilon_k\delta_{ki}p_{j\bar{l}}) - \frac{1}{n+1}\epsilon_j\delta_{ji}\{S_{k\bar{l}} + (n+1)p_{k\bar{l}}\}\\ &\quad - \frac{1}{n+1}\epsilon_k\delta_{ki}\{S_{j\bar{l}} + (n+1)p_{j\bar{l}}\}\\ &= R_{\bar{i}jk\bar{l}} - \frac{1}{n+1}(\epsilon_j\delta_{ji}S_{k\bar{l}} + \epsilon_k\delta_{ki}S_{j\bar{l}}) = G_{\bar{i}jk\bar{l}},\end{aligned}$$

so G is the same for the two projectively related connections ω and ω'. One calls G the *complex projective curvature* tensor of an indefinite Kaehler manifold (M, g).

We say that a complex hypersurface M of index $2s$ is *projective semi-symmetric* if $R(X,Y)G = 0$ for any vector fields X, Y on M. As can be easily seen (cf. Goldberg [G] and Yano and Bochner [YB] in the definite case), an indefinite Kaehler manifold M with vanishing G is of constant holomorphic sectional curvature. If M is Einstein, the projective curvature tensor coincides with the Weyl curvature tensor. Moreover, it can be easily seen that the condition $R(X,Y)R = 0$ is equivalent to $R(X,Y)G = 0$ for complex hypersurfaces M in semi-Kaehler space forms M'.

THEOREM 6.1. *Let M be an n-dimensional projective semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \le s \le n$, $t = 0$ or 1, $c \neq 0$. Then M is totally geodesic with $r = n(n+1)c$, or Einstein with $r = n^2c$, where r denotes the scalar curvature.*

Proof. Since M is projective semi-symmetric,

$$R(X,Y)G = 0, \qquad X, Y \in TM. \tag{6.3}$$

This is equivalent to

$$G_{\bar{i}jk\bar{l}m\bar{p}} - G_{\bar{i}jk\bar{l}\bar{p}m} = 0.$$

By applying the Ricci identity to G, we have

$$\sum_r \varepsilon_r(-R_{\bar{i}jr\bar{k}}G_{\bar{r}lm\bar{p}} + R_{\bar{i}jl\bar{r}}G_{\bar{k}rm\bar{p}} + R_{\bar{i}jm\bar{r}}G_{\bar{k}lr\bar{p}} - R_{\bar{i}jr\bar{p}}G_{\bar{k}lm\bar{r}}) = 0. \tag{6.4}$$

By (3.16) and (6.1) we get

$$\begin{aligned}G_{\bar{i}jk\bar{l}} &= \left\{\frac{c}{2}\varepsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{ik}\delta_{jl}) - h_{jk}\overline{h}_{il}\right\}\\ &\quad - \frac{1}{n+1}\left[\varepsilon_j\delta_{ji}\left\{\frac{(n+1)c}{2}\varepsilon_k\delta_{kl} - h_{k\bar{l}}{}^2\right\} + \varepsilon_k\delta_{ki}\left\{\frac{(n+1)c}{2}\varepsilon_j\delta_{jl} - h_{j\bar{l}}{}^2\right\}\right]\\ &= -\frac{1}{n+1}(\varepsilon_j\delta_{ji}h_{k\bar{l}}{}^2 + \varepsilon_k\delta_{ki}h_{j\bar{l}}{}^2) - h_{jk}\overline{h}_{il},\end{aligned} \tag{6.5}$$

where $h_{i\bar{j}}{}^2 = \sum_k \varepsilon_k h_{ik}\overline{h}_{kj}$ as in Section 3.

Now substituting (3.16) and (6.5) into (6.4), we obtain

$$\sum_r \varepsilon_r \big[-\big\{\tfrac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rk}+\delta_{ir}\delta_{jk}) - h_{jr}\overline{h}_{ik}\big\}\big\{\tfrac{1}{n+1}(\varepsilon_l\delta_{lr}h_{m\bar{p}}{}^2+\varepsilon_m\delta_{mr}h_{l\bar{p}}{}^2) - h_{lm}\overline{h}_{rp}\big\}$$
$$+\big\{\tfrac{c}{2}\varepsilon_{jl}(\delta_{ji}\delta_{lr}+\delta_{il}\delta_{jr}) - h_{jl}\overline{h}_{ir}\big\}\big\{\tfrac{1}{n+1}(\varepsilon_r\delta_{rk}h_{m\bar{p}}{}^2+\varepsilon_m\delta_{mk}h_{r\bar{p}}{}^2) - h_{rm}\overline{h}_{kp}\big\}$$
$$+\big\{\tfrac{c}{2}\varepsilon_{jm}(\delta_{ji}\delta_{mr}+\delta_{im}\delta_{jr}) - h_{jm}\overline{h}_{ir}\big\}\big\{\tfrac{1}{n+1}(\varepsilon_l\delta_{lk}h_{r\bar{p}}{}^2+\varepsilon_r\delta_{rk}h_{l\bar{p}}{}^2) - h_{lr}\overline{h}_{kp}\big\}$$
$$-\big\{\tfrac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rp}+\delta_{ir}\delta_{jp}) - h_{jr}\overline{h}_{ip}\big\}\big\{\tfrac{1}{n+1}(\varepsilon_l\delta_{lk}h_{m\bar{r}}{}^2+\varepsilon_m\delta_{mk}h_{l\bar{r}}{}^2) - h_{lm}\overline{h}_{kr}\big\}\big] = 0,$$

which implies that

$$\tag{6.6} \frac{c}{2(n+1)}\{\varepsilon_l\delta_{li}\varepsilon_m\delta_{mk}h_{j\bar{p}}{}^2+\varepsilon_m\delta_{mi}\varepsilon_l\delta_{lk}h_{j\bar{p}}{}^2-\varepsilon_j\delta_{jp}\varepsilon_l\delta_{lk}h_{m\bar{i}}{}^2 - \varepsilon_j\delta_{jp}\varepsilon_m\delta_{mk}h_{l\bar{i}}{}^2\}$$
$$+\frac{1}{n+1}\{-\varepsilon_m\delta_{mk}h_{jl}\overline{h}_{ip}{}^3-\varepsilon_l\delta_{lk}h_{jm}\overline{h}_{ip}{}^3+\varepsilon_l\delta_{lk}\overline{h}_{ip}h_{mj}{}^3+\varepsilon_m\delta_{mk}\overline{h}_{ip}h_{lj}{}^3\}$$
$$+\frac{c}{2}\{\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip}-\varepsilon_l\delta_{li}h_{jm}\overline{h}_{kp}-\varepsilon_m\delta_{mi}h_{lj}\overline{h}_{kp}+\varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ki}\}$$
$$+\{-\overline{h}_{ik}h_{lm}h_{j\bar{p}}{}^2+\overline{h}_{kp}h_{jl}h_{m\bar{i}}{}^2+\overline{h}_{kp}h_{jm}h_{l\bar{i}}{}^2-\overline{h}_{ip}h_{lm}h_{j\bar{k}}{}^2\}=0.$$

Setting $i=m$ in (6.6), multiplying the equation by ε_m and summing over m, we get

$$\frac{c}{2(n+1)}\{\varepsilon_l\delta_{lk}h_{j\bar{p}}{}^2+n\varepsilon_l\delta_{lk}h_{j\bar{p}}{}^2-\varepsilon_j\delta_{jp}\varepsilon_l\delta_{lk}h_2-\varepsilon_j\delta_{jp}h_{l\bar{k}}{}^2\}$$
$$+\frac{1}{n+1}\{-h_{jl}\overline{h}_{kp}{}^3+\overline{h}_{kp}h_{lj}{}^3\}+\frac{c}{2}\{\varepsilon_j\delta_{jk}h_{l\bar{p}}{}^2-h_{jl}\overline{h}_{kp}-nh_{lj}\overline{h}_{kp}+\varepsilon_j\delta_{jp}h_{l\bar{k}}{}^2\}$$
$$+\{-h_{l\bar{k}}{}^2h_{j\bar{p}}{}^2+\overline{h}_{kp}h_{jl}h_2+\overline{h}_{kp}h_{l\bar{j}}{}^3-h_{l\bar{p}}{}^2h_{j\bar{k}}{}^2\}=0,$$

where $h_{i\bar{j}}{}^3=\sum_{r,s}\varepsilon_r\varepsilon_s h_{ir}\overline{h}_{rs}h_{sj}$ as in Section 3. Setting $p=j$ in the above equation, multiplying the equation by ε_j and summing over j, we obtain

$$\tag{6.7} h_{l\bar{k}}{}^2=\frac{h_2}{n}\varepsilon_l\delta_{lk},$$

which implies that M is Einstein, because of the relation $c\neq 0$ and (3.17).

Next, we investigate the scalar curvature r of M. Since $h_{l\bar{k}}{}^2=\frac{h_2}{n}\varepsilon_l\delta_{lk}$, equation (6.6) is reduced to

$$\tag{6.8} \frac{c}{2n(n+1)}\{\varepsilon_l\delta_{li}\varepsilon_m\delta_{mk}\varepsilon_j\delta_{jp}h_2+\varepsilon_m\delta_{mi}\varepsilon_l\delta_{lk}\varepsilon_j\delta_{jp}h_2 - \varepsilon_j\delta_{jp}\varepsilon_l\delta_{lk}\varepsilon_m\delta_{mi}h_2-\varepsilon_j\delta_{jp}\varepsilon_m\delta_{mk}m\varepsilon_l\delta_{li}h_2\}$$
$$+\frac{c}{2}\{\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip}-\varepsilon_l\delta_{li}h_{jm}\overline{h}_{kp}-\varepsilon_m\delta_{mi}h_{lj}\overline{h}_{kp}+\varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ki}\}$$
$$+\frac{1}{n}\{-\overline{h}_{ik}h_{lm}\varepsilon_j\delta_{jp}h_2+\overline{h}_{kp}h_{jl}\varepsilon_m\delta_{mi}h_2+\overline{h}_{kp}h_{jm}\varepsilon_l\delta_{li}h_2-\overline{h}_{ip}h_{lm}\varepsilon_j\delta_{jk}h_2\}=0.$$

Furthermore, setting $p = j$ in (6.8), multiplying the equation by ε_j and summing over j, we get

$$(2h_2 - nc)\{(n+1)h_{lm}\overline{h}_{ik} - \varepsilon_m\delta_{mi}{h_{l\bar{k}}}^2 - \varepsilon_l\delta_{li}{h_{m\bar{k}}}^2\} = 0. \tag{6.9}$$

Since M is Einstein, h_2 is a constant. So, first consider the case where $2h_2 - nc = 0$ on M, so that the squared norm h_2 of the second fundamental form is equal to $(n/2)c$. Then in this case by (3.18) we know that M is Einstein with constant scalar curvature $r = n^2c$.

Secondly, consider the case where $2h_2 - nc \neq 0$ on M, and so (6.9) gives

$$(n+1)h_{lm}\overline{h}_{ik} - \varepsilon_m\delta_{mi}{h_{l\bar{k}}}^2 - \varepsilon_l\delta_{li}{h_{m\bar{k}}}^2 = 0, \tag{6.10}$$

from which in view of (6.7) it follows that

$$(n+1)h_{lm}\overline{h}_{ik} - \frac{1}{n}\varepsilon_m\delta_{mi}\varepsilon_l\delta_{lk}h_2 - \frac{1}{n}\varepsilon_l\delta_{li}\varepsilon_m\delta_{mk}h_2 = 0. \tag{6.11}$$

Setting $n = k$ in (6.11), multiplying the equation by ε_k and summing over k, we obtain $h_2h_{lm} = 0$. Thus $h_2 = 0$ on M, from which by (6.11) it follows that $h_{lm} = 0$ on M. Hence M is totally geodesic with scalar curvature $r = n(n+1)c$, where we have used (3.18). This completes the proof of Theorem 6.1. ∎

In particular, we consider the case where M is a projective semi-symmetric complex hypersurface in CP^{n+1}. As an application of Theorem 6.1 and Theorem B we get:

Theorem 6.2. *Let M be an n-dimensional complex hypersurface in CP^{n+1}. If M is projective semi-symmetric, then it is locally congruent to a complex quadric Q^n or to CP^n.*

Also, as in the proof of Theorem 6.1, we can easily verify the following:

Corollary 6.3. *Let M be an n-dimensional complex hypersurface in a complex hyperbolic space $H^{n+1}(c)$, $c < 0$. If M is projective semi-symmetric, then it is totally geodesic.*

Proof. When M is Einstein in the proof of Theorem 6.1 and $2h_2 = nc$ for $c < 0$, the non-negativity of the squared norm of the second fundamental form h_2 gives us a contradiction. So this case cannot occur. ∎

By using the same method as in the proof of Corollary 6.3, we get

Corollary 6.4. *Let M be an n-dimensional complex hypersurface in an indefinite complex space form $M_1^{n+1}(c)$, $c > 0$. If M is projective semi-symmetric, then it is totally geodesic.*

7. Conformal semi-symmetric complex hypersurfaces. This section is devoted to the investigation of conformal semi-symmetric hypersurfaces in complex space forms.

Let M be an n-dimensional semi-Kaehler hypersurface of index $2s$ in an $(n+1)$-dimensional semi-Kaehler space form $M' = M^{n+1}_{s+t}(c)$, $0 \le s \le n$, $t = 0$ or 1, of index $2(s+t)$ and of constant holomorphic sectional curvature c. We denote by R the Riemannian curvature tensor on M.

Recall that the conformal curvature tensor H on M has components

$$H_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}(\varepsilon_k S_{\bar{i}j}\delta_{kl} + \varepsilon_j S_{\bar{i}k}\delta_{jl} + \varepsilon_j S_{\bar{l}k}\delta_{ij} + \varepsilon_k S_{\bar{l}j}\delta_{ik}). \tag{7.1}$$

As is easily seen, H is a curvature-like tensor on M.

The hypersurface M is said to be *conformal semi-symmetric* if

$$R(X,Y)H = 0, \qquad X, Y \in TM. \tag{7.2}$$

It is easily verified that (7.2) is equivalent to

$$H_{\bar{i}jk\bar{l}m\bar{p}} - H_{\bar{i}jk\bar{l}\bar{p}m} = 0. \tag{7.3}$$

In fact, by applying the Ricci identity to H, we have

$$\sum_r \varepsilon_r(-R_{\bar{i}jr\bar{k}}H_{\bar{r}lm\bar{p}} + R_{\bar{i}jl\bar{r}}H_{\bar{k}rm\bar{p}} + R_{\bar{i}jm\bar{r}}H_{\bar{k}lr\bar{p}} - R_{\bar{i}jr\bar{p}}H_{\bar{k}lm\bar{r}}) = 0. \tag{7.4}$$

From (3.16) and (7.1) we get

$$\begin{aligned} H_{\bar{i}jk\bar{l}} = {} & \frac{1}{2(n+1)}\{\varepsilon_j(\delta_{jl}h_{k\bar{i}}{}^2 + \delta_{ji}h_{k\bar{l}}{}^2) + \varepsilon_k(\delta_{kl}h_{j\bar{i}}{}^2 + \delta_{ki}h_{j\bar{l}}{}^2)\} \\ & - h_{jk}\overline{h}_{il}. \end{aligned} \tag{7.5}$$

By substituting (3.16) and (7.5) into (7.4), we obtain

$$\begin{aligned} \sum_r \varepsilon_r \Big[& \big\{\tfrac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rk} + \delta_{ir}\delta_{jk}) - h_{jr}\overline{h}_{ik}\big\}\big(\tfrac{1}{2(n+1)}\big\{\varepsilon_l(\delta_{lp}h_{m\bar{r}}{}^2 + \delta_{lr}h_{m\bar{p}}{}^2) \\ & \qquad + \varepsilon_m(\delta_{mp}h_{l\bar{r}}{}^2 + \delta_{mr}h_{l\bar{p}}{}^2)\big\} - h_{lm}\overline{h}_{rp}\big) \\ & - \big\{\tfrac{c}{2}\varepsilon_{jl}(\delta_{ji}\delta_{lr} + \delta_{il}\delta_{jr}) - h_{jl}\overline{h}_{ir}\big\}\big(\tfrac{1}{2(n+1)}\big\{\varepsilon_r(\delta_{rp}h_{m\bar{k}}{}^2 + \delta_{rk}h_{m\bar{p}}{}^2) \\ & \qquad + \varepsilon_m(\delta_{mp}h_{r\bar{k}}{}^2 + \delta_{mk}h_{r\bar{p}}{}^2)\big\} - h_{rm}\overline{h}_{kp}\big) \\ & - \big\{\tfrac{c}{2}\varepsilon_{jm}(\delta_{ji}\delta_{mr} + \delta_{im}\delta_{jr}) - h_{jm}\overline{h}_{ir}\big\}\big(\tfrac{1}{2(n+1)}\big\{\varepsilon_l(\delta_{lp}h_{r\bar{k}}{}^2 + \delta_{lk}h_{r\bar{p}}{}^2) \\ & \qquad + \varepsilon_r(\delta_{rp}h_{l\bar{k}}{}^2 + \delta_{rk}h_{l\bar{p}}{}^2)\big\} - h_{lr}\overline{h}_{kp}\big) \\ & + \big\{\tfrac{c}{2}\varepsilon_{jr}(\delta_{ji}\delta_{rp} + \delta_{ir}\delta_{jp}) - h_{jr}\overline{h}_{ip}\big\}\big(\tfrac{1}{2(n+1)}\big\{\varepsilon_l(\delta_{lr}h_{m\bar{k}}{}^2 + \delta_{lk}h_{m\bar{r}}{}^2) \\ & \qquad + \varepsilon_m(\delta_{mr}h_{l\bar{k}}{}^2 + \delta_{mk}h_{l\bar{r}}{}^2)\big\} - h_{lm}\overline{h}_{kr}\big)\Big] = 0. \end{aligned}$$

From this, after canceling some terms in each formula on the left side, we arrive at

$$
\begin{aligned}
(7.6)\qquad & -\tfrac{c}{4(n+1)}\{\varepsilon_j\delta_{jk}(\varepsilon_l\delta_{lp}h_{m\bar{i}}{}^2+\varepsilon_l\delta_{li}h_{m\bar{p}}{}^2+\varepsilon_m\delta_{mp}h_{l\bar{i}}{}^2+\varepsilon_m\delta_{mi}h_{l\bar{p}}{}^2)\}\\
&+\tfrac{1}{2(n+1)}\{\varepsilon_l\delta_{lp}\overline{h}_{ik}h_{mj}{}^3+h_{jl}\overline{h}_{ik}h_{m\bar{p}}{}^2+\varepsilon_m\delta_{mp}\overline{h}_{ik}h_{lj}{}^3+h_{jm}\overline{h}_{ik}h_{l\bar{p}}{}^2\}\\
&+\tfrac{c}{2}\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip}-h_{j\bar{p}}{}^2h_{lm}\overline{h}_{ik}\\
&+\tfrac{c}{4(n+1)}\{\varepsilon_l\delta_{il}(\varepsilon_j\delta_{jp}h_{m\bar{k}}{}^2+\varepsilon_j\delta_{jk}h_{m\bar{p}}{}^2+\varepsilon_m\delta_{mp}h_{j\bar{k}}{}^2+\varepsilon_m\delta_{mk}h_{j\bar{p}}{}^2)\}\\
&-\tfrac{1}{2(n+1)}\{\varepsilon_m\delta_{mp}h_{jl}\overline{h}_{ik}{}^3+h_{jl}\overline{h}_{ip}h_{m\bar{k}}{}^2+\varepsilon_m\delta_{mk}h_{jl}\overline{h}_{ip}{}^3+h_{jl}\overline{h}_{ik}h_{m\bar{p}}{}^2\}\\
&-\tfrac{c}{2}\varepsilon_l\delta_{il}h_{jm}\overline{h}_{kp}+h_{m\bar{i}}{}^2h_{jl}\overline{h}_{kp}\\
&+\tfrac{c}{4(n+1)}\{\varepsilon_m\delta_{mi}(\varepsilon_l\delta_{lp}h_{j\bar{k}}{}^2+\varepsilon_l\delta_{lk}h_{j\bar{p}}{}^2+\varepsilon_j\delta_{jp}h_{l\bar{k}}{}^2+\varepsilon_j\delta_{jk}h_{l\bar{p}}{}^2)\}\\
&-\tfrac{1}{2(n+1)}\{\varepsilon_l\delta_{lp}h_{jm}\overline{h}_{ik}{}^3+h_{jm}\overline{h}_{ip}h_{l\bar{k}}{}^2+\varepsilon_l\delta_{lk}h_{jm}\overline{h}_{ip}{}^3+h_{jm}\overline{h}_{ik}h_{l\bar{p}}{}^2\}\\
&-\tfrac{c}{2}\varepsilon_m\delta_{mi}h_{lj}\overline{h}_{kp}+h_{l\bar{i}}{}^2h_{jm}\overline{h}_{kp}\\
&-\tfrac{c}{4(n+1)}\{\varepsilon_j\delta_{jp}(\varepsilon_l\delta_{li}h_{m\bar{k}}{}^2+\varepsilon_l\delta_{lk}h_{m\bar{i}}{}^2+\varepsilon_m\delta_{mi}h_{l\bar{k}}{}^2+\varepsilon_m\delta_{mk}h_{l\bar{i}}{}^2)\}\\
&+\tfrac{1}{2(n+1)}\{\varepsilon_l\delta_{lk}\overline{h}_{ip}h_{mj}{}^3+h_{jm}\overline{h}_{ip}h_{l\bar{k}}{}^2+\varepsilon_m\delta_{mk}\overline{h}_{ip}h_{jl}{}^3+h_{jl}\overline{h}_{ip}h_{m\bar{k}}{}^2\}\\
&\qquad\qquad+\tfrac{c}{2}\varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ki}-h_{j\bar{k}}{}^2h_{lm}\overline{h}_{ip}=0.
\end{aligned}
$$

From this we deduce

THEOREM 7.1. *Let M be an n-dimensional conformal semi-symmetric complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0\le s\le n$, $t=0$ or 1, $c\neq 0$. Then M is totally geodesic with $r=n(n+1)c$, or Einstein with $r=n^2c$, where r denotes the scalar curvature.*

Proof. Since $R(X,Y)H=0$ for any X,Y on M, equation (7.6) holds. Setting $p=j$ in (7.6), multiplying the equation by ε_j and summing over j, we get

$$
\begin{aligned}
-\frac{c}{4(n+1)}&\{(2n+1)(\varepsilon_l\delta_{li}h_{m\bar{k}}{}^2+\varepsilon_m\delta_{mi}h_{l\bar{k}}{}^2)+(n+1)(\varepsilon_l\delta_{lk}h_{m\bar{i}}{}^2+\varepsilon_m\delta_{mk}h_{l\bar{i}}{}^2)\\
&-(\varepsilon_l\delta_{li}\varepsilon_m\delta_{mk}+\varepsilon_m\delta_{mi}\varepsilon_l\delta_{lk})h_2\}+\frac{1}{(n+1)}(\overline{h}_{ik}h_{lm}{}^3-h_{lm}\overline{h}_{ik}{}^3)\\
&+\frac{c}{2}(n+1)h_{lm}\overline{h}_{ki}+h_{l\bar{i}}{}^2h_{m\bar{k}}{}^2-h_2h_{lm}\overline{h}_{ik}+h_{l\bar{k}}{}^2h_{m\bar{i}}{}^2-h_{lm}\overline{h}_{ki}{}^3=0.
\end{aligned}
$$

Furthermore, setting $l=k$ in the above equation, multiplying the equation by ε_k and summing over k, we get

$$
(7.7)\qquad c(nh_{m\bar{i}}{}^2-h_2\varepsilon_m\delta_{mi})=0,
$$

which implies that M is Einstein, because of $c\neq 0$ and (3.17).

Now, we investigate the scalar curvature r of M. As $h_{m\bar{i}}{}^2=(h_2/n)\varepsilon_m\delta_{mi}$, equation (7.6) is reduced to

$$
\begin{aligned}
(7.8)\qquad & nc\{\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip}-\varepsilon_l\delta_{li}h_{jm}\overline{h}_{kp}-\varepsilon_m\delta_{mi}h_{lj}\overline{h}_{kp}+\varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ki}\}\\
&+2h_2\{-\varepsilon_j\delta_{jp}h_{lm}\overline{h}_{ik}+\varepsilon_m\delta_{mi}h_{jl}\overline{h}_{kp}+\varepsilon_l\delta_{li}h_{jm}\overline{h}_{kp}-\varepsilon_j\delta_{jk}h_{lm}\overline{h}_{ip}\}=0.
\end{aligned}
$$

Setting $p = j$ in the above equation, multiplying the equation by ε_j and summing over m, we have

$$(2h_2 - nc)\{(n+1)h_{lm}\overline{h}_{ik} - \varepsilon_l\delta_{li}{h_{m\bar{k}}}^2 - \varepsilon_m\delta_{mi}{h_{l\bar{k}}}^2\} = 0. \tag{7.9}$$

Since M is Einstein, h_2 is a constant. So, first consider the case where $2h_2 - nc = 0$ on M; then the squared norm h_2 of the second fundamental form is $(n/2)c$. In this case by (3.18) we know that M is Einstein with constant scalar curvature $r = n^2c$.

Secondly, assume that $2h_2 - nc \neq 0$ on M. Then (7.9) gives

$$(n+1)h_{lm}\overline{h}_{ik} - \varepsilon_l\delta_{li}{h_{m\bar{k}}}^2 - \varepsilon_m\delta_{mi}{h_{l\bar{k}}}^2 = 0. \tag{7.10}$$

Multiplying (7.10) by $\varepsilon_k h_{kt}$ and summing over k, we get

$$(n+1)h_{lm}{h_{t\bar{i}}}^2 - \varepsilon_l\delta_{li}{h_{mt}}^3 - \varepsilon_m\delta_{mi}{h_{lt}}^3 = 0, \tag{7.11}$$

from which by (7.7) it follows that

$$(n+1)h_{lm}\varepsilon_t\delta_{ti}h_2 - \varepsilon_l\delta_{li}h_{mt}h_2 - \varepsilon_m\delta_{mi}h_{lt}h_2 = 0. \tag{7.12}$$

Setting $t = i$ in the above equation, multiplying the equation by ε_t and summing over t, we get

$$(n+2)(n-1)h_2h_{lm} = 0.$$

Thus $h_2 = 0$ on M, from which in view of (7.10) it follows that $h_{ij} = 0$ on M. In other words, M is totally geodesic with scalar curvature $r = n(n+1)c$, where we have used (3.18). ∎

Remark 7.2. If M is Einstein, then M is semi-symmetric if and only if M is conformal semi-symmetric.

In particular, we consider the case where M is a conformal semi-symmetric complex hypersurface in CP^{n+1}. As an application of Theorem 7.1 and Theorem B we obtain:

Theorem 7.3. *Let M be an n-dimensional complex hypersurface in CP^{n+1}. If M is conformal semi-symmetric, then it is locally congruent to a complex quadric Q^n or to CP^n.*

Also, as in the proof of Theorem 7.1, by using the same method as in Corollaries 5.3, 6.3 for an Einstein hypersurface M in $M^{n+1}(c)$, $c < 0$, satisfying $2h_2 = nc$, we arrive at a contradiction, because the squared norm h_2 is always non-negative. So we can easily verify the following:

Corollary 7.4. *Let M be an n-dimensional complex hypersurface in a complex hyperbolic space $H^{n+1}(c)$, $c < 0$. If M is conformal semi-symmetric, then it is totally geodesic.*

By applying the same method to space-like hypersurfaces with time-like normal direction, we can verify

COROLLARY 7.5. *Let M be an n-dimensional complex space-like hypersurface in an indefinite complex space form $M_1^{n+1}(c)$, $c > 0$. If M is conformal semi-symmetric, then it is totally geodesic.*

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